A Review of Some Work on Linear Dynamic Systems

Chong Yang [1], Denghua Zhang [2,*]

[1] College of Mathematics and Physics, North China Electric Power University, Baoding, 071003, China; yangch@ncepu.edu.cn
[2] College of Science and Technology, North China Electric Power University, Baoding, 071003, China
* Correspondence: zhangdh@ncepu.edu.cn

Abstract: Some continuous linear operators have interesting dynamic properties. The linear dynamic systems mainly studies the dynamic properties of continuous linear operators such as hypercyclicity, chaos, mixing and so on. They have important links with complex analysis, operator theory, topological theory, and differential geometry, and have a wide range of applications. In particular, hypercyclicity is a property in the case of infinite dimensional spaces. In the past twenty years, the research of linear dynamic systems has become a very active field, and there are many wonderful results. Studying linear dynamics plays a huge role in promoting the development of these disciplines. In this paper, we mainly introduce recent progress in the theory of hypercyclic and chaotic operators. We give a brief review of the wonderful research results in recent years. Some recent findings on this direction are also included.

Keywords: topologically dynamic systems; hypercyclicity; infinite dimensional spaces; chaos

1. Introduction

The dynamic system is a discipline that studies the evolution of the system over time. The movement of a system (or particle) has met by a certain condition. The differential equations have used to describe. Since the high complexity of these equations, the solution are not easy to found. At the end of the 19th century, the famous French mathematician Poincare founded the qualitative theory of differential equations, and directly studied the solution without the explicit solution of the differential equations. Thus, the focus of dynamical systems research gradually shifts from the model of the differential equation to the geometric and topological properties. At the beginning of the 20th century, Birkhoff gave the topologically dynamic system. He established a large-scale theoretical framework for the dynamic system, and gradually developed into two sets of theories, which are topologically dynamic systems and ergodic theory.

Topologically dynamic systems and ergodic theory are two directions that cross each other in the dynamic system, but at the same time, there are subtle boundaries. Topologically dynamic systems study the general continuous system, studying the most basic concepts of dynamic systems in pure meaning. The ergodic theory has further researched from the statistical point of view, and its theory is broad and heavy.

The study of the topologically dynamic systems of linear operators is a branch of the discipline that is developing rapidly in the intersection of operator theory and dynamical systems. In addition, the research hotspot has attracted much attention from Western mathematicians in recent years. Since the 1980s, the study of linear dynamic systems and the research results of related properties have close connection with ergodic theory, operator algebra, differential equations, graph theory, and so on. We refer to some references [1-9] and the recent monograph [10]. It has broad application prospects in statistics, image processing, and even chaotic cryptography. After nearly 40 years of development and improvement, the theories of linear dynamical systems have become more and more invasive in many fields. Therefore, the research of linear dynamical systems and its related fields can not only promote the interdisciplinary and fusion of different fields, but also have innovative application value.

In the topologically dynamic systems, the main research objects are the topological transitivity of continuous mapping on topological spaces. In the linear dynamic systems, the spaces studied are infinite dimensional topological vector spaces and the studied mappings are continuous linear operators. Therefore, there is a difference and a connection between the linear dynamic systems and the topologically dynamic systems. The difference is the space of the linear dynamic systems is topological vector spaces. The linear structure has given on the basis, and the continuous mapping is required to have linearity. The connection is that linear chaos is a special subclass of topologically dynamic systems. Therefore, some concepts and conclusions of linear cdynamic systems have studied in the topologically dynamic systems. We know that on a finite-dimensional space, a continuous linear operator corresponds to a finite-dimensional matrix. Therefore, the study of the behaviours of the topologically dynamic systems of continuous linear operators on finite-dimensional spaces finally transform into the study of the corresponding matrix. But there are no hypercyclic operators on a finite-dimensional topological space. So in this paper, unless otherwise stated, we always assume that the objects to study are on an infinite dimensional topological vector space and the operators are continuous and linear. Furthermore, as can be seen from the definition, since iteration is a discrete and countable process, we require that the topological vector space is separable. This paper uses *R* and *N* to represent real and non-negative integer sets, respectively.

2. The Relevant Knowledge

2.1. Topological Transitivity

In the topologically dynamic systems, the main research objects are topological transitivity of continuous mapping in topological space. We set *X* as a topological space, $T: X \to X$ is a continuous mapping. If for any two non-empty open subsets $U,\ V \subset X$, there is some $n \in N$ such that $T^n U \bigcap V \neq \varnothing$, then is topologically transitive. If $T \oplus T$ is topologically trans *T* itive on $X \oplus X$, then *T* is called weakly mixing. Further, if for any two non-empty open subsets $U,\ V \subset X$, there exists a natural number N, satisfying when $n \geq N$, $T^n U \bigcap V \neq \varnothing$, then *T* is called mixing. The three concepts exist among the following recursive relationship:

Mixing $\rightarrow$ weak mixing $\rightarrow$ topologically transitive

2.2. Hypercyclicity

The linear dynamic systems are focused on the dynamic behavior of continuous linear operators on infinite dimensional topological vector spaces. Hypercyclicity is one of the most basic concepts. It has relations with topological transitivity. In fact, an operator on a separable Fréchet space is hypercyhclic if and only if it is topologically transitive. But on a separable topological vector space, this is not true.

Definition 1. Suppose that *X* is a linear space over the field *K*=*R* or *C* endowed with a Hausdorff topology τ such that addition and scalar multiplication are continuous. Then *X* is called the topological vector space.

Let *X* be a topological vector space. The symbol *B(X)* is the set of all continuous linear operators on *X*. If $T \in B(X)$ and $x \in X$, the orbit of the vector under *T* is a set $Orb(x,T) := \{T^n x : n \in N)$.

Definition 2. Let $T \in B(X)$. If there is a vector $x \in X$ such that the orbit $Orb(x,T)$ is dense in *X*, then *T* is called hypercyclic. At this time, the vector x is called a hypercyclic vector of *T*. The set of hypercyclic vectors of *T* is denoted by *HC(T)*.

Since the definition of hypercyclicity does not require any linear structure, so it is also meaningful for any continuous mapping acting on a topological vector space. In fact, continuous mapping with dense orbits is also an important research content in topologically dynamic systems. In topologically dynamic systems, the usual spaces are topological spaces that compactness is the basic aspect. In linear dynamic systems, the spaces usually are infinite dimensional and separable. These spaces cannot be compact. Therefore, the research methods and tools in topologically dynamic systems cannot apply to linear dynamic systems. Thus, in the research of linear dynamic systems, we find many interesting phenomena and conclusions.

The concept of hypercyclicity has derived from the concept of earlier loops. Let *T* be a linear operator acting on a topological vector space *X*, if there exists a vector $x \in X$ such that the linear expansion of the orbit $Orb(T,x)$ is dense in *X*, then *T* is cyclic. This definition has closely related to the well-known subspace problem. The non-trivial closed subspace *F* has found for the linear operator *T* acting on a topological vector space. Obviously, the closure of the linear expansion of the orbit under any vector is the invariant subspace of *T*. Therefore, *T* has non-trivial invariant closed subspace. Since the closure of any orbit of operator *T* is an invariant closed subset, thus *T* has non-trivial invariant closed subsets if and only if any non-zero vector $x \in X$ is a hypercyclic vector. For the problem of invariant subspaces, despite the active efforts of many mathematicians, there are still many unresolved problems, especially for the operator case of Hilbert space.

The following is a very simple hypercycle operator.

Example 3. We set $T = 2B : l^2(N) \longrightarrow l^2(N)$ is twice the left shift operator.

$$2B(x_1, x_2, x_3, \cdots) = 2(x_2, x_3, \cdots).$$

We denote operator: $S = \frac{1}{2}F : l^2(N) \longrightarrow l^2(N)$.

$$S = \frac{1}{2}F(x_1, x_2, x_3, \cdots) := \frac{1}{2}(0, x_1, x_2, \cdots).$$

Since the set $\{y^{(k)} : k \geq 1\}$ consisting of finite sequences is a countable dense subset of $l^2(N)$. By induction, we can find a sequence $(n_k)_k$ satisfies for $k > j \geq 1$, $n_k \geq m_j + n_j, 2^{n_k} \geq 2^{n_j + k} \| y^{(k)} \|$, where m_j is the maximum index with $y^{(k)}_{m_k} \neq 0$. Then the vector $x := \sum_{k=1}^{\infty} S^{n_k} y^{(k)}$ is a hypercyclic vector for *T*.

The concept of Fréchet space generalizes the one of a Banach space.

Definition 4. Let *X* be a topological vector space with topological τ. If τ is a complete topology induced by a translation invariant metric, then *X* is called *F*-space. Further, if *X* is locally convex, then *X* is called Fréchet space.

Theorem 5. (Birkhoff's transitive theorem) Let *X* be a separable *F*-space, $T \in B(X)$. The following conditions are equivalent:

(1) T is a hypercyclic operator;

(2) T is topologically transitive.

At this time, *HC(T)* is a dense G_δ subset of *X*.

Birkhoff's transitive theorem shows that studying the hypercyclicity of continuous linear operators is also the topologically dynamic systems' behaviour.

The first hypercyclic operator is due to Birkhoff in 1929 [11]. Birkhoff showed that on the space *H(C)* of entire functions, endowed with compact-open topology, the translation operator *T*, given by $Tf(z) = f(z+1), z \in C$

is hypercyclic. Godefroy and Shapiro generalized this result. They proved that an operator on $H(C^N)$ that commutes with all translation operators and is not a scalar multiple of the identity, then this operator is hypercyclic [12]. Later, they showed that they are even frequently hypercyclic. In 2011, Madore and Martínez-Avendano gave the definition of M - hypercyclic [13]. They got that the properties of M - hypercyclic operators have a lot of similarities with hypercyclic ones. There are also many definitions and properties related to hypercyclicity, we refer to some references [14-17].

3. The Nature of Hypercyclic Vectors in Linear Dynamic Systems

3.1. Hypercycle Criterion

Hypercyclicity is so important, but in many concrete situations, it is difficult to verify for a given operator. The following is Kitai's criterion under which an operator is hypercyclic [1]. This criterion is easy to understand and useful.

Definition 6. Let X be a topological vector space and $T \in B(X)$. If there is an increasing sequence (n_i) of positive integers, two dense subsets $D_1, D_2 \subset X$, and a cluster map $S_{n_i}: D_2 \to X$ has the relationship as follows:

(1) for every $x \in D_1, T^{n_i} x \to 0;$

(2) for every $y \in D_2, S_{n_i} y \to 0;$

(3) for every $y \in D_2, T^{n_i} S_{n_i} y \to y$,

then T is called satisfying the Hypercyclic criterion.

Theorem 7. Let X be a separable F-space. If a continuous linear operator T satisfies the Hypercyclic criterion, then T is hypercyclic.

Proof: Since X is separable, we can assume that D_2 is countable and arrange it into a sequence (y_i). In addition, let $\|\cdot\|$ be an F-norm of X. Note that it is not the usual norm.

We will prove that there are sequence $(x_i) \subset X$ and $(k_i) \subset \mathrm{N}$ such that $x := \sum_{j=1}^{\infty} x_j + \sum_{j=1}^{\infty} S_{n_{k_j}} y_j$ is a hypercyclic vector of T.

In fact, this process is done by mathematical induction. Let $(\varepsilon_i) \le 2^{-i}$, for $i \ge 1$, we have

(1) $\| x_i \| \le \varepsilon_i$, $\| T^{n_{k_j}} x_i \| \le \varepsilon_i \quad (j = 1, 2, \ldots, i-1)$,

(2) $\| S_{n_{k_i}} y_i \| \le \varepsilon_i$, and $\| T^{n_{k_j}} S_{n_{k_j}} y_i \| \le \varepsilon_i \quad (j = 1, 2, \ldots, i-1)$,

(3) $\| T^{n_{k_i}} S_{n_{k_i}} y_i - y_i \| \le \varepsilon_i$,

(4) $\| T^{n_{k_i}} (\sum_{j=1}^{i-1} (x_j + S_{n_{k_j}} y_j) + x_i \| \le \varepsilon_i$.

By above, we get that the series defining x converges. For any $i \in N$

$$
\begin{aligned}
&\| T^{n_{k_i}} x - y_i \| \\
&= \| T^{n_{k_i}} (\sum_{j<i} (x_j + S_{n_{k_j}} y_i) + x_i) + T^{n_{k_i}} S_{n_{k_i}} y_i - y_i \\
&\quad + \sum_{j \ge i} T^{n_{k_i}} x_j + \sum_{j \ge i} T^{n_{k_i}} x_i S_{n_{k_j}} y_j \| \\
&\le l\varepsilon_l + \varepsilon_l + \sum_{j>l} 2^{-j}.
\end{aligned}
$$

Since (y_i) is dense in X, so the above defined point x is hypercyclic for T.

Costakis and Sambarino have proved that if T satisfies the Hypercyclic criterion for a syndetic sequence, then T is mixing [18]. Zhang and Dong [19] discussed the relevant properties under the strong operator topology of the operator space and gave some important conclusions. Tajmouati and Berrag gave a subspace-hypercyclicity criterion and characterized properties of subspace-hypercyclic (resp. subspace-supercyclic) C_0-semigroup [20].

3.2. Linear Chaos

The concept of chaos firstly appeared in [21] of Li and Yorke in 1975. Since then the theory and applications of chaos have been extensively developed (see [22-25]). For a metric space (X,d) and a continuous mapping $f: X \to X$, if there is an uncountable set $\Gamma \subseteq X$ makes it satisfy for any different $x, y \in \Gamma$

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0,$$

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0,$$

then f is called Li-Yorke chaos, where the uncountable set Γ is called the scrambled set.

Since then, for the needs of research, Various extensions of definitions of chaos were presented. For a continuous linear operator T acting on a topological vector space X, if $n \in N$ is present such that $T^n x = x$, then x is a periodic point of T. If a operator is hypercyclic and has a dense periodic points, the operator is chaotic in the sense of Devaney. It is well known that if T is a hypercyclic operator acting on a complex Banach space, then λT with $|\lambda| = 1$ is also hypercyclic, but Bayart and Bermúdez prove the existence of chaotic (in the sense of Devaney) semigroup $\{\lambda T^n : |\lambda| = 1, n \ge 0\}$ on a separable Hilbert space, but every λT with $|\lambda| = 1$ is not chaotic [26].

Distributional chaos was introduced by Schweizer and Smítal in [27] as a natural extension of Li-Yorke chaos. They got that if f is a mapping from a compact interval into itself, the existence of distributionally chaotic pairs implies positive topological entropy for f. That is to say, in the case of interval maps, distributional chaos, ω - chaos, positive topological entropy are all equivalent properties. About the relations of these chaos, we refer to [28-30] for more details.

Distributional chaos plays a very strong chaotic behavior. Martínez-Giménez et al. have shown that there are uniform distributed chaos but not super-cyclic left shift operators on the Köthe space that composed of infinite dimensional matrices [31]. They later proved in the literature hypercyclic and mixed properties are not sufficient conditions for distributed chaos [32].

Bermúdez et al. obtained that if is a continuous operator on Banach spaces, then it is Li- Yorke chaotic and admits an irregular vector are equivalent [23]. Bernardes et al. got the similar results on Fréchet spaces [24]. They also gave descriptions about the dense set of irrgular vctors. Yin et al. gave sufficient conditions under which the direct sum of countable linear operators on Banach spaces is Li-Yorke chaotic [25].

4. Conclusion

This paper briefly expounds the research background of linear dynamic systems, the development status at home and abroad. Some basic concepts of linear dynamic systems have briefly introduced, and some new results have combined. The examples deepen the understanding of these concepts and guidelines. We mainly consider the hypercyclicity and give a detailed description. At the same time, we extend these results to a more general case. We know that for the study of the dynamic properties of operators on finite-dimensional spaces, there are many good methods. But on infinite dimensional spaces, many techniques have lost their effectiveness, which requires us to find new methods or theories to make research.

Acknowledgments

This work supported by the Fundamental Research Funds for the Central Universities (Grant No. 9161017007), the Fundamental Research Funds for the Central Universities (Grant No. 9161718004).

References

[1] C. Kitai. Invariant closed sets for linear operators. Ph.D. thesis, University of Toronto, 1982.

[2] K.G. Grosse-Erdmann. Universal families and hypercyclic operators. *Bull. Amer. Math. Soc.* **1999**, 36(3): 345-381.

[3] F. Bayart and S. Grivaux, Frequently hypercyclic operators. *Trans. Amer. Math. Soc.* **2006**, 358: 5083–5117.

[4] N.C. Bernardes, A. Bonilla, V. Müller and A. Peris. Distributional chaos for linear operators. *J. Funct. Anal.,* **2013**, 265 (9): 2143–2163.

[5] S. Grivaux and É. Matheron. Invariant measures for frequently hypercyclic operators. *Adv. Math.,* **2014**, 265: 371-427.

[6] S. Bartoll, F. Martínez-Giménez and A. Peris. Operators with the specification property. *J. Math. Anal. Appl.,* **2016**, 436(1): 478-488.

[7] R.R. Jiménez-Munguía, V. Galán, F. Martínez-Giménez and A. Peris, Chaos for linear fractional transformations of shifts. *Topol. Appl.,* **2016**, 203: 84-90.

[8] M. Kostić. Hypercyclic and topologically mixing properties of degenerate multi-term fractional differential equations. *Differential Equations and Dynamical Systems*, **2016**, 24(4): 475-498.

[9] N.C. Bernardes, A. Peris and F. Rodenas. Set-valued chaos in linear dynamics. *Integral Equations and Operator Theory*, **2017**, 88(4): 51-463.

[10] K.G. Grosse-Erdmann and A. Peris. Linear Chaos, Springer, London, 2011.

[11] G.D. Birkhoff. Démonstration d'un théorème élémentaire sur les fonctions entières. *C. R. Acad. Sci. Paris.* **1929**, 189: 473-475.

[12] G. Godefroy and J.H. Shapiro. Operators with dense, invariant, cyclic vector manifolds. *J. Funct. Anal.,* **1991**, 98: 229-269.

[13] B.F. Madore and R.A. Martinez-Avendano. Subspace hypercyclicity. *J. Math. Anal. Appl.,* **2011**, 373: 502-511.

[14] E.M. Mangino and M. Murillo-Arcila. Frequently hypercyclic translation semigroups. *Stud. Math.,* **2014**, 227(3): 219-238.

[15] J. Bès, O. Martin and R. Sanders. Weighted shifts and disjoint hypercyclicity. *J. Operator Theory*, **2014**, 72(1): 15-40.

[16] Q. Menet. Hereditarily hypercyclic subspaces. *J. Operator Theory*, **2015**, 73(2): 385-405.

[17] Z. Yin, Y. Chen and S. He. Disjoint Hypercyclicity and Topological Entropy of Composition Operators. *Int. J. Bifurcat. Chaos,* **2018**, 28(04): 1850053(1-13).

[18] G. Costakis and M. Sambarino. Topologically mixing hypercyclic operators. *P. Am. Math. Soc.* **2003**, 132(2): 385-389.

[19] L. Zhang and X.T. Dong, Mixing properties in the operator algebra using Hilbert-Schmidt operators. *J. Comput. Anal. Appl.,* **2015**, 18: 383–389.

[20] A. Tajmouati and M.E. Berrag. On subspace-hypercyclic and supercyclic semigroup. *International Journal of Pure and Applied Mathematics,* **2017**, 116(4): 819-827.

[21] T.Y. Li and J.A. Yorke. Period three implies chaos. *Amer. Math. Monthly,* **1975**, 82: 985-992.

[22] E. Akin and S. Kolyada. Li-Yorke sensitivity. *Nonlinearity*, **2003**, 16: 1421-1433.

[23] T. Bermúdez, A. Bonilla, F. Martmez-Giménez and A. Peris. Li-Yorke and distributionally chaotic operators. *J. Math. Anal. Appl.,* **2011**, 37(1): 83-93.

[24] N.C. Bernardes, A. Bonilla, V. Müller and A. Peris. Li-Yorke chaos in linear dynamics. *Ergodic Theory and Dynamical Systems,* **2015**, 35(1): 1723-1749.

[25] Z. Yin, S. He and Y. Huang. On Li-Yorke and distributionally chaotic direct sum operators. *Topology Appl.,* **2018**, 239(15): 35-45.

[26] F. Bayart and T. Bermúdez. Semigroups of chaotic operators. *Bull. London Math. Soc.* **2009**, 42: 823-830.

[27] B. Schweizer and J. Smítal. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Am. Math. Soc.* 1994, 344(2): 737-754.

[28] B. Schweizer, A. Sklar and J. Smítal. Distributional (and other) chaos and its measurement. *Real. Anal. Exch.,* **2001/2002**, 27: 495-524.

[29] F. Balibrea, J. Smital and M. Stefankova, The three versions of distributional chaos. *Chaos Soliton. Fract.,* **2005**, 23(5): 1581-1583.

[30] L. Luo and B. Hou. Some remarks on distributional chaos for bounded linear operators. *Turk. J. Math.,* **2015**, 39(2): 251-258.

[31] F. Martínez-Giménez, P. Oprocha and A. Peris. Distributional chaos for backward shifts. *J. Math. Anal. Appl.,* **2009**, 351: 607-615.

[32] F. Martínez-Giménez, P. Oprocha and A. Peris. Distributional chaos for operators with full scrambled sets. *Math Z.,* **2013**, 274: 603–612.

Chong Yang was born in Linyi City, Shandong province, China, in 1979. He received M.S. degree in Applied Mathematics from the Shanxi Normal University in 2005, and

received Ph.D. Degree in Applied Mathematics from the Beijing University of Technology in 2016.

Since 2005, he has been a lecture in College of Mathematics and Physics, North China Electric Power University. His research interest includes the theory of operator, the system of topological dynamics.

Denghua Zhang was born in Suizhou City, Hubei province, China, in 1980. She received M.S. degree in Applied Mathematics from the Shanxi Normal University in 2005.

Since 2005, she has been a lecture in College of Science and Technology, North China Electric Power University. Her research interest includes the theory of operator, the system of topological dynamics.